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A SUBMODEL OF HELICAL MOTIONS IN GAS DYNAMICS†

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An invariant submodel, constructed in a subalgebra from the sum of a rotation and a displacement [1], is considered within the framework of the PODMODELI program. A group classification is carried out and the optimal system of subalgebras, which is compared with the optimal system of the basic model, is calculated. Furthermore, the system of equations of the submodel is reduced to a symmetric hyperbolic form. Simple solutions of this system, with pressure and density which depend solely on time, are considered. The characteristics, the characteristic conoid, trajectories and strong discontinuities are calculated for these simple solutions. The necessary conditions for the existence of a solution without a singularity on the axis are derived.

1. EQUATIONS OF THE SUBMODEL AND THEIR SYMMETRIZATION

The system of gas-dynamic equations in cylindrical coordinates

$$\begin{aligned}
&\rho d\mathbf{U} + \nabla p = \mathbf{f}, \quad A^{-1}dp + U_x + V_r + r^{-1}W_\theta = -r^{-1}V \\
&d\rho = \rho(U_x + V_r + r^{-1}W_\theta + r^{-1}V) = 0 \quad \text{when } dS = 0 \\
&(\nabla = (\partial_x, \partial_r, r^{-1}\partial_\theta), \quad \mathbf{f} = (0, \rho W^2, -\rho VW), \quad A = \rho c^2, \quad c^2 = \partial f/\partial\rho \\
&d = \partial_t + U\partial_x + U\partial_r + r^{-1}\partial_\theta)
\end{aligned} \tag{1.1}$$

is considered, where $\mathbf{U} = (U, V, W)$ is the velocity, p is the pressure, ρ is the density, S is the entropy and $p = f(\rho, S)$ is the equation of state. System (1.1) with an arbitrary function $A(p, \rho)$ admits of 11 transformations and a continuous parametric group of transformations with the Lie algebra L_{11} [1].

A submodel of helical motions is constructed as an invariant solution using the one-dimensional subalgebra $H = \{X_1 + X_7\} \subset L_{11}$, where $X_1 = \partial_x$ is the operator of displacement with respect to the variable x and $X_7 = \partial_\theta$ is the operator of rotation about the x axis, written in the cylindrical variables x, r, θ which are related to the Cartesian coordinates by the equalities

$$\begin{aligned}
&x_1 = x, \quad x_2 = y = r\cos\theta, \quad x_3 = z = r\sin\theta \\
&u_1 = U, \quad u_2 = V\cos\theta - W\sin\theta, \quad u_3 = V\sin\theta + W\cos\theta
\end{aligned} \tag{1.2}$$

An invariant solution is sought in the form

$$\begin{aligned}
&U = U(t, r, s), \quad V = V(t, r, s), \quad W = W(t, r, s) \\
&p = p(t, r, s), \quad \rho = \rho(t, r, s), \quad s = x - \theta
\end{aligned}$$

The equalities $r = \text{const}$ and $s = \text{const}$ correspond to a helix.

By the transformation of the invariants

$$u = V, \quad \upsilon = U - r^{-1}W, \quad w = W \tag{1.3}$$

the factor system reduces to one of the evolutionary type

$$\rho(\mathbf{u}_t + u\mathbf{u}_r + \upsilon\mathbf{u}_s) + p_r = \rho\mathbf{a}, \quad A^{-1}(p_t + up_r + \upsilon p_s) + u_r + \upsilon_s = -r^{-1}u$$

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$$\rho_t + u\rho_r + \upsilon\rho_s + \rho(u_r + \upsilon_s) = -r^{-1}u\rho \quad \text{or} \quad S_t + uS_r + \upsilon S_s = 0 \tag{1.4}$$

$$(\mathbf{u} = (u, \upsilon, w), \quad \mathbf{a} = (a^1, a^2, a^3) = (r^{-1}w^2, 2r^{-2}uw, -r^{-1}uw))$$

In order to study the correctness of the Cauchy problem for system (1.4), it is necessary to reduce it to a symmetric form [2, p. 70]. To do this, the equation for the entropy is chosen and a linear transformation of the velocities is carried out using the formulae

$$\upsilon^i = b^i_j u^j (u^1 = u,\ u^2 = \upsilon,\ u^3 = w), \quad u^j = c^j_i \upsilon^i, \quad b^i_j c^j_k = \delta^i_k$$

System (1.4) is transformed into the system which, in matrix form, is

$$A^t \mathbf{q}_t + A^1 \mathbf{q}_r + A^2 \mathbf{q}_s = \mathbf{D}, \quad \mathbf{q} = (\upsilon^1, \upsilon^2, \upsilon^3, p, S)^T \tag{1.5}$$

$$A^t = \operatorname{diag}\{\rho, \rho, \rho, A^{-1}, 1\}, \quad \mathbf{D} = (d_1, d_2, d_3, d_4, 0)^T$$

$$A^k = \begin{Vmatrix} \rho c^k_i \upsilon^i & 0 & 0 & b^1_k b_k + b^1_3 b_{k+2} & 0 \\ 0 & \rho c^k_i \upsilon^i & 0 & b^2_k b_k + b^2_3 b_{k+2} & 0 \\ 0 & 0 & \rho c^k_i \upsilon^i & b^3_k b_k + b^3_3 b_{k+2} & 0 \\ c^k_1 & c^k_2 & c^k_3 & A^{-1} c^k_i \upsilon^i & 0 \\ 0 & 0 & 0 & 0 & c^k_i \upsilon^i \end{Vmatrix}, \quad k = 1, 2$$

$$b_1 = 1, \quad b_2 = 1 + r^{-2}, \quad b_3 = 0, \quad b_4 = -r^{-1}, \quad d_4 = -ur^{-1} - \upsilon^i (c^1_{ir} + c^2_{is})$$

$$d_i = \rho b^i_j a^i + \rho c^j_k \upsilon^k (\partial_t + c^n_m \upsilon^m \partial_{x^n}) b^i_j, \quad x^1 = r, \quad x^2 = s$$

In the case of symmetric matrices A^k, it is required that the six equalities

$$c^k_i = b^i_k b_k + b^i_3 b_{k+2}, \quad k = 1, 2; \quad i = 1, 2, 3 \tag{1.6}$$

for determining the nine elements of the matrix $B = (b^i_j)$ must be satisfied.

If one introduces the angles α_{ij} between the vectors $\mathbf{b}_i = (b^1_i, b^2_i, b^3_i)$ and $\mathbf{b}_j$, it then follows from (1.6)

$$\alpha_{13} = \alpha_{12} = \pi/2; \quad |\mathbf{b}_1| = 1, \quad |\mathbf{b}_2| = r(1 + r^2)^{-1/2} \sin^{-1}\alpha_{23}, \quad |\mathbf{b}_3| = (1 + r^2)^{1/2} \operatorname{ctg} \alpha_{23}$$

Specification of the angle α_{23} and the directions of the vectors $\mathbf{b}_1$ and $\mathbf{b}_3$ (or $\mathbf{b}_2$) with the conditions $\mathbf{b}_1 \cdot \mathbf{b}_2 = \mathbf{b}_1 \cdot \mathbf{b}_3 = 0$ defines the matrix B.

For example, let

$$\alpha_{23} = \pi/4, \quad \mathbf{b}_1 = (1, 0, 0)^T, \quad \mathbf{b}_3 = (0, 0, (1 + r^2)^{1/2})^T$$

$$\mathbf{b}_2 = (0, r(1 + r^2)^{-1/2},\ r(1 + r^2)^{-1/2})^T$$

then

$$c^1_1 = 1, \quad c^1_2 = c^1_3 = c^2_1 = c^2_3 = 0, \quad c^2_2 = (1 + r^{-2})^{1/2}$$

Here

$$\mathbf{D} = (\beta r^2 \rho (\upsilon^2 - \upsilon^3)^2,\ \ \beta\rho\upsilon^1(\upsilon^2 + 2\upsilon^3),\ \ \beta\rho\upsilon^1(2\upsilon^2(1 + r^2) + \upsilon^3(1 - 2r^2)),\ \ -r^{-1}\upsilon^1, 0)^T$$

$$\beta = r^{-1}(1 + r^2)^{-1}$$

2. SIMPLE SOLUTIONS

The system of equations (1.4) resembles the system of equations of plane gas dynamics for which a constant solution is a simple solution which depends on five arbitrary constants and is specified in the whole space for all times. It would also be desirable to have a solution with analogous properties in the case of system (1.4). It is proposed that we seek a solution with a pressure and density which depend solely on time $p = p(t)$, $\rho = \rho(t)$. In this case, (1.4) becomes the overdetermined system of equations

$$A(p, \rho)d\rho = \rho dp \tag{2.1}$$
$$\mathbf{u}_t + u\mathbf{u}_r + \upsilon\mathbf{u}_s = \mathbf{a}, \quad u_r + \upsilon_s + r^{-1}u + \rho^{-1}\rho' = 0$$

The first equation of the system determines the function $p(t)$ if the function $\rho(t)$ is specified. System (2.1) defines isentropic flows and it is necessary to investigate it for compatibility. The equations of the system are initially integrated in Lagrangian variables.

Remark. System (2.1) follows the symmetry of the basic system of equations of gas dynamics, the normalization factor of the algebra H in the algebra L_1: ∂_t, ∂_s, $t\partial_s + \partial_\upsilon$. In addition to this, the extension operators $r\partial_r + u\partial_u + w\partial_w$, $\rho\partial_\rho$ are permitted.

The change to Lagrangian variables is defined by the system of ordinary differential equations

$$\partial_t r = u(t, r, s), \quad \partial_t s = \upsilon(t, r, s); \quad r|_{t=0} = \xi, \quad s|_{t=0} = \eta \tag{2.2}$$

The solution of (2.2) $r = r(t, \xi, \eta)$, $s = s(t, \xi, \eta)$ defines the change to the variables ξ, η (t is a parameter) is the Jacobian of the transformation is non-zero $r_\xi s_\eta - r_\eta s_\xi \neq 0$.

In Lagrangian variables, all the equations of system (2.1) are integrated with respect to the variable t

$$r^2 = \xi^2 + \alpha^2 t^2 + 2\alpha_1 t, \quad s = \eta + \beta t - \operatorname{arctg}[\gamma t(\alpha_1 t + \xi^2)^{-1}] \tag{2.3}$$

$$\alpha^2\xi^2 = \gamma^2 + \alpha_1^2 \tag{2.4}$$

$$u = r^{-1}(\alpha^2 t + \alpha_1), \quad \upsilon = \beta - \gamma r^{-2}, \quad w = \gamma r^{-1} \tag{2.5}$$

$$r(r_\xi s_\eta - r_\eta s_\xi) = \rho^{-1}J \tag{2.6}$$

The quantities $\alpha \neq 0$, α_1, $\gamma \neq 0$, β, $J \neq 0$ depend on $\xi \neq 0$, η. In (2.6), the initial data are not taken into account since they will be subsequently changed. On substituting (2.3) into (2.6), we obtain an equality from which a rational form of the function

$$\rho = P_3(t)/P_5(t), \quad P_3 = S_2t^2 + S_1t + S_0, \quad P_5 = T_5t^5 + T_4t^4 + T_3t^3 + T_2t^2 + T_1t + T_0$$

is determined.

With such a function $\rho(t)$, equality (2.6) has a free variable t which occurs in a rational manner. Equating the coefficients accompanying the different powers of t to zero we obtain eight equations for the functions α, α_1, β and the constants T_i, S_i.

Equivalence in the set of possible solutions is introduced using the transformations which are permitted by system (2.1) and have been noted in the remark. They form a group G_5 (invariant variables are not indicated): (1) $t' = t + t_0$; (2) $s' = s + s_0$: (3) $s' = s + ct$, $\upsilon' = \upsilon + c$; (4) $r' = ar$, $u' = au$, $w' = aw$; (5) $\rho' = b\rho$. These transformations are extended to the coefficients of the solution (2.3)

$$\alpha' = a^{-1}\alpha, \quad \alpha_1' = a^{-2}\alpha^2 t_0 + \alpha_1 a^{-2}, \quad \xi'^2 = a^{-2}\xi^2 + \alpha^2 a^{-2} t_0^2 + 2\alpha_1 a^{-2} t_0$$

$$J' = Ja^{-2}b^{-1}, \quad \beta' = \beta - c, \quad \gamma' = \gamma a^{-2}$$

$$\eta' = \eta - s_0 + (\beta - c)t_0 - \operatorname{arctg}[\gamma t_0(\alpha_1 t_0 + \xi^2)^{-1}]$$

and, in this case, equality (2.4) remains invariant. The parameters of the transformations t_0, s_0, a, b, c are functions of ξ, η and, in this case, the initial data for problem (2.2) change. The extended transformations form a group G_5, the Lie algebra of which is defined by the basis set of operators

$$\partial_\eta, \ \partial_\beta, \ J\partial_J, \ \alpha^2\partial_{\alpha_1} + \alpha_1\xi^{-1}\partial_\xi + (\beta - \gamma\xi^{-2})\partial_\eta, \quad \alpha\partial_\alpha + 2\alpha_1\partial_{\alpha_1} + \xi\partial_\xi + 2\gamma\partial_\gamma$$

The invariants of the group G_5 are: $I = \gamma\alpha^{-2}(\alpha_1^2 - \alpha^2\xi^2)\gamma^{-2} = -1$. The value of the second invariant is taken by virtue of equality (2.4).

The new parameters of the solution $\zeta = \alpha_1\alpha^{-2}$, I are introduced

$$r^2 = \alpha^2[I^2 + (t+\zeta)^2], \quad s = \eta + \beta t - \operatorname{arctg}[It(\zeta^2 + \zeta t + I^2)^{-1}] \tag{2.7}$$

$$u = \alpha^2 r^{-1}(t+\zeta), \quad \upsilon = \beta - I\alpha^2 r^{-2}, \quad w = I\alpha^2 r^{-1} \tag{2.8}$$

The equivalence transformations of the invariant parameters take the form

$$\eta' = \eta - s_0 + (\beta - c)t_0 - \operatorname{arctg}[It_0(\zeta t_0 + \zeta^2 + I^2)^{-1}], \quad \alpha' = \alpha a^{-1}$$

$$\zeta' = \zeta + t_0 a^{-1}, \quad \beta' = \beta - c$$

In the space of the parameters α, ζ, β, η, the group acts transitively and, hence, any values of the parameters can be obtained from the fixed values, for example, $\alpha = 1$, $\zeta = 0$, $\beta = 0$, $\eta = 0$.

Formulae (2.7) define the change to Lagrangian variables if not all of the parameters I, α, ζ, β, η are fixed. If the magnitude of the parameter I is arbitrary, it is possible to fix three parameters from α, ζ, β, η and, moreover, by three methods.

The case $\zeta = \beta = \eta = 0$ gives the solution

$$\rho = t^{-1}, \quad u = rt^{-1}\sin^2 s, \quad \upsilon = t^{-1}\sin s\cos s, \quad w = -rt^{-1}\sin s\cos s \tag{2.9}$$

The case $\alpha = 1$, $\beta = \eta = 0$, after a displacement with respect to t, gives the solution

$$\rho = 1, \quad u = tr^{-1}, \quad \upsilon = -r^{-2}(r^2 - t^2)^{1/2}, \quad w = r^{-1}(r^2 - t^2)^{1/2} \tag{2.10}$$

which is invariant with respect to the operator ∂_y.

The case $\alpha = 1$, $\zeta = \eta = 0$ gives the solution

$$\begin{aligned} &\rho = t^{-1}, \quad u = tr^{-1} \\ &\upsilon = st^{-1} - r^{-2}(r^2 - t^2)^{1/2} + t^{-1}\operatorname{arctg}[t(r^2 - t^2)^{-1/2}] \\ &w = r^{-1}(r^2 - t^2)^{1/2} \end{aligned} \tag{2.11}$$

which is invariant with respect to the operator $t\partial_y + \partial_\upsilon$.

The case $\alpha = 1$, $\zeta = \beta = 0$ leads to (2.10).

Let I be a fixed function of the parameters α, ζ, β, η. Then, only two parameters can be fixed.

The case $\alpha = 1$, $\zeta = 0$, $I = I(\beta, f)$ leads to the solution

$$\begin{aligned} &\rho = (T_1 t + T_0)^{-1}, \quad u = tr^{-1} \\ &\upsilon = -r^{-2}(r^2 - t^2)^{1/2} + (T_1 t + T_0)^{-1} T_1\{s + \operatorname{arctg}[t(r^2 - t^2)^{-1/2}] + g(r^2 - t^2)\} \\ &w = r^{-1}(r^2 - t^2)^{1/2} \end{aligned}$$

where $g(\lambda)$ is an arbitrary function.

Remark. The quantity $I = I(T_0\beta + T_1 f)$ is expressed in terms of an arbitrary function, and it can therefore be taken as an independent parameter in the Lagrangian transform and the solution can thereby be reduced to the formulae which have been considered earlier. This remark refers to cases when the magnitude of I is determined with a functional arbitrariness.

The case $\alpha = 1$, $\beta = 0$, $I = I(\zeta, f)$ leads to the solution

$$\begin{aligned} &\rho = (T_1 t + T_0)^{-1}, \quad u = (t+\zeta)r^{-1}, \quad \upsilon = -r^2[r^2 - (t+\zeta)^2]^{1/2} \\ &w = r^{-1}[r^2 - (t+\zeta)^2]^{1/2} \end{aligned}$$

The function $\zeta = \zeta(t, r, s)$ is determined from the equality

$$g(K) = s + \operatorname{arctg}[t(r^2 - (t+\zeta)^2)^{1/2})(r^2 - t^2 + t\zeta)^{-1}] + \\ + T_0 T_1^{1/2} \int [K + 2T_0\zeta(K + 2T_0\zeta - T_1\zeta)^{1/2}]^{-1} \zeta d\zeta$$

where $g(K)$ is an arbitrary function, $K = T_1(r^2 - t^2 - 2t\zeta) - 2T_0\zeta$.

The case $\alpha = 1, \eta = 0. I = I(\zeta, \beta)$ gives the solution

$$\rho = t^{-1}(t + T_0)^{-1}, \quad u = (t+\zeta)r^{-1}$$
$$\upsilon = st^{-1} - r^{-2}[r^2 - (t+\zeta)^2]^{1/2} + t^{-1}\operatorname{arctg}[t(r^2 - (t+\zeta)^2)^{1/2}(r^2 - t^2 - t\xi)^{-1}]$$
$$w = r^{-1}[r^2 - (t+\zeta)^2]^{1/2}$$

The function $\zeta = \zeta(t, r, s)$ is determined from the inequality

$$g(R^2) = st^{-1} + t^{-1}\operatorname{arctg}[t(R^2 - (\zeta - T_0)^2)^{1/2})(R^2 - T_0^2 + t\zeta + 2T_0\zeta)^{-1}] + \\ + \frac{1}{2}R^{-1}T_0^{-2}(T_0 + R)^2 \operatorname{arctg}[(T_0 - R)(T_0 + R)^{-1}(R + T_0 - \zeta)^{1/2}(R - T_0 + \zeta)^{-1/2}] + \\ + \frac{1}{2}T_0^{-1}\arccos[R^{-1}(\zeta - T_0)]$$

where $g(R^2)$ is an arbitrary function, $R^2 = r^2 - t^2 + T_0^2 - 2\zeta(t + T_0)$.

The case $\zeta = \beta = 0, I = I(\alpha, f)$ leads to the solution

$$\rho = (t^2 + T_0)^{-1}, \quad u = t(r^2 + I_0)r^{-1}(t^2 + T_0)^{-1} \tag{2.12}$$
$$\upsilon = -r^{-2}(r^2 + I_0)^{1/2}(T_0 r^2 - I_0 t^2)^{1/2}(t^2 + T_0)^{-1}$$
$$w = r^{-1}(r^2 + I_0)^{1/2}(T_0 r^2 - I_0 t^2)^{1/2}(t^2 + T_0)^{-1}$$

The case $\zeta = \eta = 0, I = I(\alpha, \beta)$ leads to the solution

$$\rho = t^{-1}(t^2 + \tau^2)^{-1}, \quad u = tr(I^2 + t^2)^{-1}$$
$$\upsilon = st^{-1} + t^{-1}\operatorname{arctg}(tI^{-1}) - I(I^2 + t^2)^{-1}, \quad w = rI(I^2 + t^2)^{-1}$$

The function $I = I(t, r, s)$ is determined from the equality

$$(I - \tau)(I + \tau)^{-1}\exp[2\tau(st^{-1} + t^{-1}\operatorname{arctg}(tI^{-1}))] = g(r^2(I^2 - \tau^2)(I^2 + \tau^2)^{-1})$$

where τ is arbitrary constant and $g(\lambda)$ is an arbitrary function.

The case $\beta = \eta = 0, I = I(\alpha, \zeta)$ leads to the solution

$$\rho = t^{-1}(T_1 t + T_0)^{-1}, \quad u = r(t+\zeta)[I^2 + (t+\zeta)^2]^{-1}$$
$$\upsilon = -I[I^2 + (t+\zeta)^2]^{-1}, \quad w = Ir[I^2 + (t+\zeta)^2]^{-1}$$

where $\zeta = -1/2t \pm (1/4t^2 - I^2 - It \operatorname{ctg} s)^{1/2}$ and the function $I = I(t, r, s)$ is determined with a functional arbitrariness from the differential equation

$$T_1\zeta\alpha I_\alpha + [T_1(I^2 - \zeta^2) + T_0\zeta]I_\zeta = I(T_0 - 2T_1\zeta)$$

It remains to consider the possibility of $w = 0$ in (2.5). The solution of system (2.1) in Lagrangian variables takes the form

$$r = \alpha t + \xi, \quad s = \beta t + \eta; \quad u = \alpha, \quad \upsilon = \beta, \quad w = 0$$

Transformations (1)–(5), extended to the coefficients, form a transitive group

$$\alpha' = \alpha a^{-1}, \quad \beta' = \beta - c, \quad \xi' = a^{-1}(\xi + \alpha t_0), \quad \eta' = \eta + t_0(\beta - c) - s_0, \quad J' = Jb^{-1}a^{-2}$$

All the coefficients $\alpha = 1, \xi = \beta = \eta = 0, J = 1$ can be fixed by these transformations. In the Lagrangian substitution, two parameters must remain arbitrary. Only the following consistent cases are obtained.

The case $\beta = \xi = 0$

$$\rho = t^{-2}, \quad u = rt^{-1}, \quad \upsilon = w = 0 \tag{2.13}$$

The case $\eta = \xi = 0$

$$\rho = t^{-3}, \quad u = rt^{-1}, \quad \upsilon = st^{-1}, \quad w = 0 \tag{2.14}$$

Thus, solutions of system (2.1) which allow of a Lagrangian substitution are reduced by equivalence transformations to the simplest solutions (2.9)–(2.14). The solutions (2.9)–(2.12) allow of an increase in the constant parameters up to five with the transforms of the permitted group (1)–(5). An explicit formula for the pressure is obtained in the case of a polytropic gas $A = \gamma p$ from the first equation of system (2.1): $p = B\rho^{-\gamma}$.

$$(2.9) \Rightarrow u = r(t+t_0)^{-1}\sin^2(s - \upsilon_0 t - s_0) \tag{2.15}$$

$$\upsilon = \upsilon_0 + \tfrac{1}{2}(t+t_0)^{-1}\sin 2(s - \upsilon_0 t - s_0)$$

$$w = -\tfrac{1}{2}r(t+t_0)^{-1}\sin 2(s - \upsilon_0 t - s_0), \quad \rho = \rho_0(t+t_0)^{-1}, \quad p = p_0(t+t_0)^{-\gamma}$$

$$(2.10) \Rightarrow u = w_0^2 r^{-1}(t+t_0), \quad \upsilon = \upsilon_0 - w_0 r^{-2}[r^2 - w_0^2(t+t_0)^2]^{1/2} \tag{2.16}$$

$$w = w_0 r^{-1}[r^2 - w_0^2(t+t_0)^2]^{1/2}, \quad \rho = \rho_0, \quad p = p_0$$

$$(2.11) \Rightarrow u = w_0^2 r^{-1}(t+t_0) \tag{2.17}$$

$$\upsilon = (s+s_0)(t+t_0)^{-1} - w_0 r^{-2}[r^2 - w_0^2(t+t_0)^2]^{1/2} +$$

$$+(t+t_0)^{-1}\operatorname{arctg}\{w_0(t+t_0)[r^2 - w_0^2(t+t_0)^2]^{-1/2}\}$$

$$w = w_0 r^{-1}[r^2 - w_0^2(t+t_0)^2]^{1/2}, \quad \rho = \rho_0(t+t_0)^{-1}, \quad p = p_0(t+t_0)^{-\gamma}$$

$$(2.12) \Rightarrow u = r^{-1}t(r^2 + I_0)(t^2 + T_0)^{-1} \tag{2.18}$$

$$\upsilon = \upsilon_0 - r^{-2}(r^2 + I_0)^{1/2}(T_0 r^2 - I_0 t^2)^{1/2}(t^2 + T_0)^{-1}$$

$$w = r^{-1}(r^2 + I_0)^{1/2}(T_0 r^2 - I_0 t^2)^{1/2}(t^2 + T_0)^{-1}$$

$$\rho = \rho_0(t^2 + T_0)^{-1}, \quad p = p_0(t^2 + T_0)^{-\gamma}$$

When $I_0 = 0$, solution (2.18), after the displacement $t_1 = t + \upsilon_0 w_0^{-1}$, has the form

$$u = w_0^2 r t_1(t_1^2 w_0^2 + 1)^{-1}, \quad \upsilon = u_0 - w_0(t_1^2 w_0^2 + 1)^{-1} \tag{2.19}$$

$$w = w_0 r(t_1^2 w_0^2 + 1)^{-1}, \quad \rho = \rho_0(t_1^2 w_0^2 + 1)^{-1}$$

$$p = p_0(t_1^2 w_0^2 + 1)^{-\gamma}, \quad S = S_0, \quad c^2 = \gamma p_0 \rho_0^{-1}(t_1^2 w_0^2 + 1)^{1-\gamma}$$

3. CHARACTERISTICS FOR SIMPLE SOLUTIONS

In the case of system (1.2), the characteristics $g(x, r, \theta) = \text{const}$ for the solution are determined from the equations [2, p. 60]

$$C_0\colon\ g_t + Ug_x + Vg_r + r^{-1}Wg_\theta = 0 \quad \text{(threefold)}$$

$$C_\pm\colon\ g_t + Ug_x + Vg_r + r^{-1}Wg_\theta \pm cQ = 0, \quad Q = (g_x^2 + g_r^2 + r^{-2}g_\theta^2)^{1/2}$$

The bicharacteristics satisfy the system of ordinary differential equations

$$C_0\colon\ d_t x = U, \quad d_t r = V, \quad r d_t\theta = W$$

$$C_\pm\colon\ d_t x = U \pm cg_x Q^{-1}, \quad d_t r = V \pm cg_r Q^{-1}, \quad r d_t \theta = W \pm r^{-1} c g_\theta Q^{-1}$$

$$-d_t g_x = U_x g_x + V_x g_r + r^{-1} W_x g_\theta \pm c_x Q, \quad -d_t g_\theta = U_\theta g_x + V_\theta g_r + r^{-1} W_\theta g_\theta \pm c_\theta Q$$

$$-d_t g_r = U_r g_x + V_r g_r (r^{-1}W)_r g_\theta \pm c_r Q \mp c r^{-3} g_\theta^2 Q^{-1}$$

In the case of system (1.4), there are three invariant characteristics

$$C_0\colon\ h_t + uh_r + \upsilon h_s = 0 \quad \text{(threefold)}$$

$$C_\pm\colon\ h_t + uh_r + \upsilon h_s \pm cq = 0, \quad q = (h_r^2 + (1 + r^{-2})h_s^2)^{1/2}$$

The bicharacteristics are defined by the equations

$$C_0\colon\ d_t r = u, \quad d_t s = \upsilon$$

$$C_\pm\colon\ d_t r = u \pm ch_r q^{-1}, \quad d_s s = \upsilon \pm ch_s (1 + r^{-2}) q^{-1}, \quad d_t h_s = -u_s h_r - \upsilon_s h_s \mp c_s q$$

$$d_t h_r = -u_r h_r - \upsilon_r h_s \mp c_r q \pm c r^{-3} h_s^2 q^{-1}$$

For the simple solution (2.19), the following expressions are obtained for the invariant quantities: for the bicharacteristic C_0

$$r = r_0 (t_1^2 w_0^2 + 1)^{1/2}, \quad s = x_0 - \theta_0 + u_0 t_1 - \operatorname{arctg}(w_0 t_1) \tag{3.1}$$

and the characteristic surface has the form

$$h = \Phi(r(t_1^2 w_0^2 + 1)^{-1/2}, \quad s - u_0 t_1 + \operatorname{arctg}(w_0 t_1)) = C$$

The representation

$$\begin{aligned} r &= (t_1^2 w_0^2 + 1)^{1/2} G^{1/2}(t_1) \\ s &= x_0 - \theta_0 + u_0 t - \operatorname{arctg}(w_0 t_1) + \lambda(\gamma p_0 \rho_0^{-1})^{1/2} \int_0^{t_1} (z^2 w_0^2 + 1)^{-\gamma/2} \times \\ &\times [1 + \lambda^2 (z^2 w_0^2 + 1)]^{-1/2} [G^{-1}(z) + 1 + z^2 w_0^2] dz \end{aligned} \tag{3.2}$$

where

$$G(t) = r_0^2 + 2(r_0^2 - \lambda^2)^{1/2} g(t) + g^2(t)$$

$$g(t) = (\gamma p_0 \rho_0^{-1})^{1/2} \int_0^t (z^2 w_0^2 + 1)^{-\gamma/2} [1 - \lambda^2 (z^2 w_0^2 + 1)]^{-1/2} dz$$

holds for the bicharacteristic C_4.

For fixed x_0, r_0, θ_0, formulae (3.2) define a parametric representation of a characteristic conoid with parameter λ. The vertex of the conoid is obtained when $t = 0$. For small t_1 close to the vertex of the characteristic conoid, its intersection by a plane $t_1 = \text{const}$ is a circle

$$R^2 + S^2 = \gamma p_0 \rho_0^{-1} t_1^2 r_0^{-2} \tag{3.3}$$

and, moreover, the centre of the circle moves along the trajectory (3.1).

When $0 < \gamma < 2$, the integral $g(\infty)$ converges while the integral in the expression for s diverges. Hence, when $t_1 \to \infty$, the oval (3.2) is elongated along the S axis.

The bicharacteristics C_0 for (1.2) are

$$x = x_0 + u_0 t_1, \quad \theta = \theta_0 + \operatorname{arctg}(w_0 t_1), \quad r = r_0(1 + t_1^2 w_0^2)^{1/2} \tag{3.4}$$

The projection of the line (3.4) onto $\mathbb{R}^3(x, r, \theta)$ is a straight line which is represented by the equalities $x = x_0 + u_0 w_0^{-1} r_0^{-1} z$, $y = r_0$ in the Cartesian system of coordinates $y = r\cos(\theta - \theta_0)$, $z = r\sin(\theta - \theta_0)$.

The bicharacteristics C_+ for system (1.2) in solution (2.19) are defined by the equalities

$$r = r_0(1 + w_0^2 t_1^2)^{1/2}[1 + c_0^2 r_0^{-2}(\lambda^2 - 1)g^2 + 2c_0 r_0^{-1}(\lambda^2 - 1)^{1/2} g(1 - \mu^2 r_0^{-2})^{1/2}]^{1/2}$$

where

$$g = \int_0^{t_1} (z^2 w_0^2 + 1)^{-\gamma/2}(\lambda^2 + z^2 w_0^2)^{-1/2} dz$$

$$x = x_0 + u_0 t_1 + c_0 \int_0^{t_1} (z^2 w_0^2 + 1)^{1-\gamma/2}(\lambda^2 + z^2 w_0^2)^{-1/2} dz$$

$$\theta = \theta_0 + \operatorname{arctg}(w_0 t_1) + \mu(\lambda^2 - 1)^{1/2} c_0 \int_0^{t_1} (z^2 w_0^2 + 1)^{1-\gamma/2}(\lambda^2 + z^2 w_0^2)^{-1/2} r^{-2} dz$$

If the parameters λ and μ are eliminated from these equalities, then a hypersurface in the space of the variables t_1, x, r, θ is obtained which determines the characteristic conoid. When $t_1 \to 0$, the conoid is defined by the equalities

$$(x - x_0 - u_0 t_1)^2 + (r - r_0(1 + w_0^2 t_1^2)^{1/2})^2 + (\theta - \theta_0 - \operatorname{arctg}(w_0 t_1))^2 = c_0^2 t_1^2$$

The intersection of the conoid by a plane t_1 = const is a closed surface in the space $\mathbb{R}^3$ and, in this case, a point on the trajectory lies within this closed surface. The intersection of the surface by a plane θ = const is a part of a circle ($r > 0$) with centre at the point $(x_0 + u_0 t_1, r_0(1 + w_0^2 t_1^2)^{1/2})$ and radius $(c_0^2 t_1^2 - (\theta - \theta_0 - \operatorname{arctg}(w_0 t_1))^2)^{1/2}$.

If t_1 is small, a real circle exists for angles θ which only slightly differ from $\theta_0 + \operatorname{arctg}(w_0 t_1)$.

The following expressions are obtained for the solution (2.15): for the bicharacteristic C_0

$$x = x_0 + \upsilon_0 t_1, \quad r = r_0(1 + C^2 t_1^2)^{1/2}, \quad \theta = \theta_0 + \operatorname{arctg}(C t_1)$$

where r_0 and C are constants and the trajectories are the straight lines $y = r_0$, $\upsilon_0 z = \pm r_0 C(x - x_0)$, where x, y, z are Cartesian coordinates. The characteristics are defined by the equalities $x = \upsilon_0 t_1 + \psi(y, z t_1^{-1})$, where ψ is an arbitrary function.

4. STRONG DISCONTINUITIES FOR SIMPLE SOLUTIONS

The invariant surface and the velocity of motion of the invariant surface in the direction of the normal are described, in the variables of the helical motion, by the formulae $G(t, r, s) = 0$, $D_n = -G_t(G_r^2 + G_s^2(1 + r^{-2}))^{-1/2}$.

The equations of a non-removable discontinuity are:
a contact discontinuity

$$[p] = 0, \quad \omega_i = G_t + u_i G_r + \upsilon_i G_s = 0, \quad i = 1, 2 \tag{4.1}$$

a shock wave

$$[\rho\omega] = 0, \quad [p + \rho\omega^2] = 0, \quad H(\rho_2, p_2; \rho_1, p_1) = 0 \tag{4.2}$$

where $\omega = (G_t + uG_r + \upsilon G_s)(G_r^2 + G_s^2(1 + r^{-2}))^{-1/2}$, $H(\rho, p; \rho_1, p_1) = \varepsilon(\rho^{-1}, p) - \varepsilon(\rho_1^{-1}, p_1) + 1/2(\rho^{-1} - \rho_1^{-1})(p + p_1)$ is the Hugoniot function and ε is the internal energy. For a polytropic gas, the

Hugoniot adiabatic curve $H = 0$ takes the form

$$p_2 p_1^{-1} = [(\gamma+1)\rho_2 - (\gamma-1)\rho_1][(\gamma+1)\rho_1 - (\gamma-1)\rho_2]^{-1} \tag{4.3}$$

In the case of a non-invariant surface of a non-removable discontinuity $G(t, x, r, \theta) = 0$, the relative velocity is equal to

$$\omega = (G_t + UG_x + VG_r + Wr^{-1}G_\theta^2)(G_x^2 + G_r^2 + r^{-2}G_\theta^2)^{-1/2}$$

For solutions (2.18), there can only be shock waves when $T_0 = 0, \gamma = 2$. In this case, the relations

$$[\rho_0(I_0 - N)] = 0, \quad [p_0 + \rho_0(I_0 - N)^2] = 0, \quad p_{02}p_{01}^{-1} = (3\rho_{02} - \rho_{01})(3\rho_{01} - \rho_{02})^{-1}$$

are satisfied and the invariant surface of the shock wave is a moving cylinder in $\mathbb{R}^3$

$$r = \ln^{-1}(kT^{-1}) \text{ when } N = 0; \quad \tau(Kt^{2\tau} + 1)(1 - Kt^{2\tau})^{-1} \text{ when } N = -\tau^2$$

$$\tau\mathrm{tg}(\tau\ln(tT^{-1})) \text{ when } N = \tau^2; \text{ where } N \text{ and } K \text{ are constants}$$

The contact discontinuity for solution (2.18) is non-invariant. It is possible when $T_0 = 0$ and is defined by the equalities $x = \upsilon_0 t + x_0$; $[p_0] = [\upsilon_0] = 0$.

There can only be an invariant contact discontinuity for solutions (2.19) on a cylindrical surface $r = r_0[1 + (w_0 t + \upsilon_0)^2]^{1/2}$ with the conditions $[p_0] = [\upsilon_0] = [w_0] = 0$, $[u_0] = 0$, $[u_0] \neq 0$.

Only a non-invariant non-removable discontinuity is possible in the case of the solutions (2.15). A contact discontinuity is defined by the equalities $x = \upsilon_0 t_1 + x_0$; $[p_0] = [\upsilon_0] = [t_0] = 0$, $[s_0] \neq 0$. A shock wave is only possible when $\gamma = 1$: $x = Nt + x_0$, $[t_0] = 0$, $[\rho_0(\upsilon_0 - N)] = 0$, $[p_0 + \rho_0(\upsilon_0 - N)^2] = 0$, $p_{02}\rho_{02}^{-1} = p_{01}\rho_{01}^{-1}$.

For solutions (2.16), an invariant contact discontinuity is a cylinder $r = [w^2{}_0(t + t_0)^2 + r^2{}_0]^{1/2}$ on which the conditions $[t_0] = [w_0] = [p_0] = 0$, $[\upsilon_0] \neq 0$ are satisfied. There is no invariant shock wave for the set of solutions (2.16). The plane $x = \upsilon_0 t - x_0$ with the conditions $[\upsilon_0] = [p_0] = 0$ is a non-invariant contact discontinuity. A non-invariant shock wave exists which is defined by the plane $x = Nt + x_0$ with the conditions $[\rho_0(\upsilon_0 - N)] = 0$, $[p_0 + \rho_0(\upsilon_0 - N)^2] = 0$ and (4.3) with zero subscripts. The invariant shock wave has the form $r = N(t + t_0)$ with the conditions $[\rho_0(w_0^2 - N] = 0$, $[Np_0 + \rho_0(w_0^2 - N)^2] = 0$ and (4.3) with zero subscripts.

In the case of solutions (2.17), the relations $[t_0] = [w_0] = [p_0] = 0$ are satisfied at the contact discontinuity. Its equation is $r^2 = w^2{}_0 t_1^2 + r_0^2$ for an invariant contact discontinuity and $\theta = \mathrm{arctg}[w_0 t_1(r^2 - w_0^2 t_1^2)^{-1/2} + \psi(r^2 - w_0^2 t_1^2)$ for a non-invariant contact discontinuity, where ψ is an arbitrary function. A shock wave can only be invariant when $\gamma = 1$

$$r = Nt_1, \quad [\rho_0(w_0^2 - N)] = 0, \quad [Np_0 + \rho_0(w_0^2 - N)^2] = 0, \quad p_{02}\rho_{02}^{-1} = p_{01}\rho_{01}^{-1}$$

5. GROUP CLASSIFICATION

System (1.4) with the arbitrary element $A = A(\rho, p)$ has the following equivalence transformations: (1) $p' = a_1 p + a_2$, $\rho' = a_1\rho$, $A' = a_1 A$; (2) $p' = -p$, $\rho' = -\rho$, $A' = -A$; (3) $t' = a_3 t$, $u' = a_3^{-1}u$, $\upsilon' = a_3^{-1}\upsilon$, $w' = a_3^{-1}w$, $p' = a_3^{-2}p$, $A' = a_3^{-2}P$.

The result of the group classification of system (1.4) is presented in Table 1 [11].

Explanation of Table 1. The kernel $m = 1$ occurs in all 12 Lie algebras and r is the dimension of the algebra. All the algebras are factor algebras of the normalizers of a subalgebra of H and, in the corresponding algebras with special factors A [1, Table 1], with respect to H. For $m = 10$, $A = \pm\rho$ in the general case. The plus sign is taken on the basis of physical considerations since $A = \rho c^2 > 0$, $\rho > 0$. The functions g, encountered in the table, are arbitrary: $Y_4 + Y + 2p\partial_p$, $Z_\gamma = (1 - \gamma)Y + 2\rho\partial_\rho + 2\gamma p\partial_p$, $Y_5 = Y + 2p\partial_p$, $Y = t\partial_t - u\partial_u - \upsilon\partial_\upsilon - w\partial_w$.

6. THE OPTIMAL SYSTEM OF SUBALGEBRAS FOR A PERMISSIBLE LIE ALGEBRA IN THE CASE OF A GENERAL EQUATION OF STATE

A Lie algebra $L_3 = \{Y_1, Y_2, Y_3\}$ which is permitted by system (1.4) has a single non-zero commutator $[Y_3, Y_2] = Y_1$. There is a two-parameter family of non-trivial isomorphisms of the algebra: A_2: $x'^1 =$

Table 1

m	A	Operators	r
1	$g(\rho, p)$	$\{Y_1 = \partial_y, Y_2 = t\partial_y + \partial_\upsilon, Y_3 = \partial_t\}$ – kernel	3
2	$pg(p\rho^{-\gamma})$	Z_γ	4
3	$pg(p\rho^{-1})$	Z_1	4
4	$g(\rho)$	Z_0	4
5	$pg(\rho)$	Y_5	4
6	γp	Z_0, Z_1	5
7	$g(\rho e^{-p})$	$Z_0 + 2\partial_p$	4
8	$g(\rho)$	∂_p	4
9	$\gamma\rho^\gamma$	Z_γ, ∂_p	5
10	ρ	Z_1, ∂_p	5
11	1	Z_0, ∂_p	5
12	0	$Z_0, \rho g'(p)\partial_\rho + g(p)\partial_p$	∞

Table 2

r	N	Basis	Normalizer	Subalgebra of L_{11}	Subalgebra [1, Table 6]
3	1	1, 2, 3	= 3,1	1, 4, 7, 10	4.4°
2	1	1,2 + α3	3,1	1,7,4 + α10	~3.9° when $\alpha \neq 0$ or 3.11 when $\alpha = 0$
2	2	1.3	3.1	1, 7, 10	3.2°°
1	1	2 + α3	2,1	1 + 7,4 + α10	~2.7 when $\alpha \neq 0$ or 2.10 when $\alpha = 0$
1	2	1	3,1	1,7	2.9°
1	3	3	2,2	1 + 7, 10	2.6

$x^1 - a_2x^3, A_3: x'^1 = x^1 + a_3x^2$. Using these, the optimal system of subalgebras is obtained which reduces to the normalized system in Table 2 [1], where the numbers of operators forming the subalgebras are shown. The normalizers are the subalgebras r.N. Subalgebras from L_{11}, which are similar to subalgebras from the main table of subalgebras of the algebra L_{11} [1, Table 6] correspond to the subalgebras r.N. The "~" sign denotes similarity while the "=" sign signifies the autonormalization character of a subalgebra [1]. Invariant and partially invariant solutions of rank 1 and 2 [3, pp. 247, 282] will be considered for the subalgebras indicated in the final column of Table 2. Here, only the irreducible, partially invariant solution of rank 1 of a defect 1, constructed in the whole Lie algebra L_3 which is permitted by the submodel (1.4), is shown.

The integrals

$$S(p,\rho) = S_0 \quad (p = f(\rho, S_0)), \quad u^2 + M(\rho) = u_0^2 - Br^{-1}, \quad r\rho u = DC(C - \int u^{-1}dr)^{-1}$$

hold, where

$$M = 2\int_0^\rho \rho^{-1} f_\rho(\rho, S_0)d\rho$$

S_0, B, C and D are constants and the remaining functions are defined by the formulae $\upsilon = [C\varphi(t = \int u^{-1}dr) - s - BCr^{-2} - 2B\int (r^{-3}\int u^{-1}dr)dr](C - \int u^{-1}dr)^{-1}$, and $w = Br^{-1}\varphi(s)$ is an arbitrary function.

When $C \to \infty$, $\varphi = \upsilon_0$, an invariant solution, constructed in the subalgebra $\{Y_1, Y_3\}$, is obtained.

7. NECESSARY CONDITIONS FOR THE EXISTENCE OF A SOLUTION WITHOUT A SINGULARITY ON THE AXIS

When $r = 0$, the submodel (1.4) can have a singularity. Here, it will be shown when the solution can be represented by series in the neighbourhood of the axis $r = 0$ (summation over all $k \geqslant 0$)

$$u=\sum u_k r^k, \quad \upsilon=\sum \upsilon_k r^k, \quad w=\sum w_k r^k, \quad \rho=\sum \rho_k r^k, \quad p=\sum p_k r^k, \quad A=\sum (k!)^{-1} A_k r^k$$

$$A_k = D_r^k A(p,\rho)|_{r=0} = k!(A_\rho^0 \rho_k + A_p^0 p_k + A_{\rho p}^0(\rho_{n-1}p_1+\rho_1 p_{n-1})+A_{pp}^0 p_{n-1}p_1)+\ldots$$

Substitution of the series into system (1.4) and comparison of the coefficients accompanying the same powers of the variable r gives

$$\begin{aligned}
&\sum_{j=0}^{k-1} u_{jt}\rho_{k-j} + \sum_{i=1}^{k} \rho_{k-1} \sum_{j=1}^{i} j u_j u_{i-j} + \sum_{i=0}^{k-1} \rho_{k-1-i} \sum_{j=0}^{i-1} u_{js}\upsilon_{i-j} + \\
&+kp_k - \sum_{i=0}^{k} \rho_{k-1} \sum_{j=0}^{i} w_j w_{i-j} = 0 \\
&\sum_{i=2}^{k} \rho_{k-i}\left(\upsilon_{i-2t} + \sum_{j=1}^{i-1} u_{i-1-j} j\upsilon_j + \sum_{j=0}^{i-2} \upsilon_{js}\upsilon_{i-2-j}\right) + \\
&+p_{ks} + p_{k-2s} - 2\sum_{i=0}^{k} \rho_{k-i} \sum_{j=0}^{i} u_j w_{i-j} = 0 \\
&\sum_{i=1}^{k} \rho_{k-i}\left(w_{i-1t} + \sum_{j=1}^{i} u_{i-j} j w_j + \sum_{j=0}^{i-1} w_{js}\upsilon_{i-1-j}\right) - \\
&-p_{ks} + \sum_{i=0}^{k} \rho_{k-i} \sum_{j=0}^{i} u_j w_{i-j} = 0 \\
&\rho_{k-1t} + \sum_{j=1}^{k} u_{k-j} j\rho_j + \sum_{j=0}^{k-1} \rho_{js}\upsilon_{k-1-j} + \sum_{j=0}^{k} \rho_{k-j}(u_j(1+j)+\upsilon_{j-1s}) = 0 \\
&p_{k-1t} + \sum_{j=1}^{k} u_{k-j} j p_j + \sum_{j=0}^{k-1} p_{js}\upsilon_{k-1-j} + \sum_{j=0}^{k} ((k-j)!)^{-1} A_{k-j}(u_j(1+j)+\upsilon_{j-1s}) = 0
\end{aligned} \tag{7.1}$$

The physical meaning of the helical motions lies in the fact that

$$u_0 = w_0 = 0, \quad \upsilon_{0s} = \rho_{0s} = p_{0s} = 0, \quad \rho_0 \neq 0$$

When $k = 0$, the equations obtained become identical.
When $k = 1$

$$p_1 = 0, \quad u_1 = -\tfrac{1}{2}\rho_0^{-1}\rho_0', \quad p_0 = f(\rho_0, S_0)$$

are determined, where f is a general solution of the equation $\rho_0 dp_0 = A_0 d\rho_0$ and S_0 is a constant of integration.
When $k = 2$, the quantities

$$\begin{aligned}
&w_1 = -\upsilon_0 - a'(s_1), \quad s_1 = s - \int \upsilon_0 dt \\
&\rho_1 = B(\rho_0)b(s_1), \quad p_2 = s(\upsilon_0\rho_0' - \rho_0\upsilon_0') - \rho_0' a + m(t) \\
&B = \rho_0^{3/2} \exp(-\int A_p(\rho_0, f(\rho_0))A^{-1}(\rho_0, f(\rho_0))d\rho_0) \\
&u_2 = -\tfrac{1}{3}\upsilon_{1s} + \tfrac{1}{3} B\rho_0'\rho_0^{-1} A_p^0 A_0^{-1} b
\end{aligned}$$

are determined from system (7.1).
The equality remains

$$\begin{aligned}
&\tfrac{1}{4}\rho_0^{-1}\rho_0'^2 - \tfrac{1}{2}(\ln \rho_0)'' Bb(s_1) - \rho_0(\upsilon_0 - a'(s_1))^2 + \\
&+2s(\upsilon_0\rho_0' - \rho_0\upsilon_0') - 2\rho_0' a(s_1) + 2m(t) = 0
\end{aligned} \tag{7.2}$$

which it is necessary to investigate for compatibility.

An equality is obtained after differentiating (7.2) twice with respect to s and once with respect to t from which only the possibilities follow

$$1)\ a''=0,\ b''=0;\ \ 2)\ a''=0,\ [(\ln\rho_0)''B(\rho_0)\rho_0^{-1}]'=0$$
$$3)\ [b''a''^{-1}]'=\upsilon_0'=0;\ \ 4)\ [(\ln\rho_0)''B(\rho_0)\rho_0^{-1}]'=\upsilon_0'=0$$

The solutions of Eq. (7.2) in each of the cases are

Case 1

$$a=\alpha_1 s_1+\alpha_0,\quad b=\beta_1 s_1+\beta_0,\quad \upsilon_0=V_0\rho_0+\alpha_1-\frac{1}{4}\beta_1\rho_0\int B(\rho_0)\rho_0^{-1}(\ln\rho_0)''dt$$
$$m=\alpha_0\rho_0'-(\upsilon_0\rho_0'-\rho_0\upsilon_0')\int\upsilon_0 dt+\frac{1}{2}(\upsilon_0-\beta_1)^2-\frac{1}{8}\rho_0^{-1}\rho_0'^2+\frac{1}{4}\beta_0(\ln\rho_0)''$$

where α_0, α_1, β_0, β_0, V_0 are constants and $\rho_0(t)$ is an arbitrary function.

Case 2

$$a=\alpha_1 s_1+\alpha_0,\quad \rho_0=C_0e^{Ct},\quad \upsilon_0=\rho_0V_0+\alpha_1$$
$$m=\rho_0\left(C\alpha_0-\frac{1}{8}C^2-C\alpha_1^2t-\alpha_1V_0\rho_0+\frac{1}{2}V_0^2\rho_0^2\right)$$

where α_0, α_1, C_0, C, V_0 are constants and $b(s_1)$ is an arbitrary function.

Case 3

$$a=\upsilon_0 s_1+\frac{1}{2}C_1s_1^2,\quad b=\beta_0+\frac{1}{8}CC_1s_1^2,\quad m=\upsilon_0t\rho_0'-\frac{1}{8}\rho_0^{-1}\rho_0'^2-C^{-1}\beta_0\rho_0\left(\ln\rho_0+\frac{1}{4}C_1\right)$$

where $s_1=s-\upsilon_0t$, $C\neq 0$, C_1, β_0, υ_0 are constants and the function $\rho_0(t)$ is determined from the equation

$$CB(\rho_0)(\ln\rho_0)''+\rho_0(4\ln\rho_0+C_1)=0$$

Case 4

$$\rho_0=C_0e^{Ct},\quad m=-CC_0\upsilon_0^2e^{Ct}(t+t_0),\quad a=-\frac{1}{2}Cs_1^2+\alpha_1s_1+\frac{1}{8}C^2-t_0\upsilon_0^2-\frac{1}{2}C^{-1}(\upsilon_0-\alpha_1)^2$$

where C_0, C, α_1, υ_0, t_0 are constants and $b(s_1)$ is an arbitrary function.

When $k=3$, the quantities $p_3=2/3\rho_0(a'-\upsilon_0)w_2+1/6(\ln\rho_0)''\rho_2-1/6\rho'_0\upsilon_{1s}+1/9\rho_0\upsilon_0\upsilon_{1ss}+1/9Bb\upsilon_{1ts}+L_1$ are determined from system (7.1) and a linear system of equations is obtained for w_2, ρ_2, υ_1.

The function u_3 is determined from the fourth equation of (7.1) after which υ_2 is found in the following step

$$-4\rho_0u_3=\rho_0\upsilon_{2s}+\rho_{2t}+u_2\rho_1+\rho_{1s}\upsilon_1+\rho_{2s}\upsilon_0+4\rho_2u_1+\rho_1(3u_2+\upsilon_{1s})$$

and this equality is used to derive an equation for υ_2.

It is proved that, if p_{k-1}, u_i, υ_{i-2}, w_i, ρ_i, p_i, $i<k-1$ and u_{k-1} are determined in terms of υ_{k-2} at the $(k-1)$th step, then p_k, u_k are found in terms of υ_{k-1} at the kth step and a system of differential equations is also obtained for finding υ_{k-2}, w_{k-1}, ρ_{k-1}.

If $\rho_0=C_0e^{Ct}$, the system splits and an equation for ρ_{k-1} separates out. If, in addition to this, $a''=0$, then the equation for u_k is integrated and a single equation is obtained for υ_{k-2}.

So, cases 1–4 define the necessary conditions for the existence of a solution of system (1.4) without a singularity on the $r=0$ axis.

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